Geometry of Random Surfaces

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April 27, 2019

Random surfaces in moduli space

- $\mathcal{M}_g$ = moduli space of compact Riemann surfaces of genus g
- Fenchel-Nielsen coordinates on $\mathcal{M}_g$: given by length of curves in a pair of pants decomposition $\ell_1, ..., \ell_{3g-3}$, and twist parameters $\tau_1, ..., \tau_{3g-3}$ that indicate how to glue along the boundaries of the pairs of pants

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- Question: if we pick a random surface from $\mathcal{M}_g$ according to the WP volume, what does it look like geometrically?
  - shortest geodesic? $\geq C$ with high probability asymptotically
  - diameter? $\leq C \log g$ with probability 1 asymptotically
  - Cheeger constant? $\geq C$ with probability 1 asymptotically [Mirzakhani]

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- Conjecture [Brooks-Makover, Mirzakhani, Guth-Parlier-Young]: discrete measure is a good asymptotic approximation for the WP volume on $\mathcal{M}_g$

References

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